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Some remarks on the differential - geometric approach to supermanifolds*

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Abstract. *We analyze the category of GH^∞ supermanifolds recently introduced by Rogers and show that these supermanifolds do not have a good graded tangent bundle, and that a natural definition of super vector bundle is not possible within that category. However, any GH^∞ supermanifold can be turned into a supermanifold of a new category (that we call a $\mathcal{G}$-supermanifold) which is well-behaved, and is a particular case of a supermanifold à la Rothstein.*

1. INTRODUCTION

Supermanifolds and graded manifolds were originally introduced to provide a mathematical setting for physical theories whose geometric substratum incorporates «anticommuting objects». Examples of theories of such a kind are geometrical quantization [1], classical (i.e. non-quantum) supergravity [2, 3] and

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supersymmetric field theory [2, 4] and the theory of supersymmetric integrable systems [5]. The recent introduction of differential-topological methods in supersymmetric field theory (basically in connection with the anomaly problem) and in superstring theories indicates that one needs a better understanding of the global geometry of these new structures – namely, supermanifolds and graded manifolds. For instance, it has been shown that some anomalies of super Yang-Mills theories can be computed in terms of suitable cohomologies defined on supermanifolds [6].

Graded manifolds were the first ones to receive a rigorous mathematical treatment, starting with Berezin and Lĕites [7] and Kostant [1]. Basically, a graded manifold consists in a sheaf $\mathscr{A}$ of $\mathbf{Z}_2$-graded commutative algebras over a smooth manifold X; in a sense, one does not enlarge the set of space-time points, but rather the set of «observables», namely one replaces the structure sheaf that X has as a smooth manifold with a bigger sheaf $\mathscr{A}$. So the study of graded manifolds needs sheaf theory – an area of mathematics that theoretical physicists have begun to be familiar with only recently. Besides, they have a poor topological structure; so to say, they have trivial topology *«in the odd directions»*. This implies that graded manifolds are uninteresting as far as cohomology is concerned; basically, all cohomology is contained in the base smooth manifold [1].

In order to obtain a theory with stronger geometric contents, Rogers (partly following De Witt [8]) developed a different approach, where the set of points in enlarged by modelling the manifold not on a euclidean space, but rather on a generalization of it, where the real numbers are replaced by a Grassmann algebra $B_L = \Lambda(\mathbf{R}^L)$ [9]. The objects obtained in this way will be here called supermanifolds. Actually, the real point is the choice of the category of transition functions used to model the manifold. The original choice by Rogers, the so-called «G^∞ functions», is not a good one (unless one takes the limit $L \to \infty$; in this connection see Ref. 10). Indeed, as Boyer and Gitler [11] pointed out, the resulting structure sheaf has a sheaf of derivations which is not locally free, which prevents one from using local coordinates to get local descriptions of vector fields and from giving a sensible notion of graded tangent space.

Recently two remedies have been proposed. Rothstein gives a new definition of supermanifold, which is again in terms of sheaves and generalizes the category of graded manifolds (see Ref. 12 and Section 5 of this paper). Rothstein's supermanifolds are in a sense intermediate between graded manifolds and G^∞-supermanifolds, in that the relevant structure sheaf is larger than the structure sheaf of the underlying smooth manifold, but (in general) not so much as in the graded manifold case. Even though it is not known whether any G^∞-supermanifold can be turned into a Rothstein supermanifold, this is certainly possible if the topology of the supermanifold is not too complicated (for details see Ref. [12]).

A different solution has been put forward by Rogers, who proposes a modification of the definition of G^∞ functions, actually by introducing a new type of morphisms, that she calls «GH^∞ functions» [13]. According to Rogers' claim, the sheaf $Der\ \mathcal{GH}$ of graded derivations of the sheaf $\mathcal{GH}$ of GH^∞ functions on a supermanifold M is locally free, as we show in Section 3. However, the sheaf $\mathcal{GH}$ is improperly behaved in other respects. The main point is that if $\mathcal{GH}_x$ is the stalk of $\mathcal{GH}$ at $x \in M$, namely, the graded algebra of germs of GH^∞ functions at x, and $\mathcal{M}_x$ is the ideal of germs which vanish when evaluated at x, it possible that the quotient modules $\mathcal{GH}_x/\mathcal{M}_x$ (which are the sets of the values taken by GH^∞ functions at x) considered for different x's are not isomorphic. This implies for instance that a GH^∞ functions is not a section of a suitable trivial bundle on M in any sensible way, and that a graded tangent bundle with a standard fibre does not exist; indeed its fibre at $x \in M$ ought to be isomorphic to $(\mathcal{GH}_x/\mathcal{M}_x)^{m+n}$ if M has dimension (m, n).

In this paper we analyze these peculiarities of GH^∞ functions (Section 3). Moreover, in Section 4 we show that the sheaf $\mathcal{GH}$ on «flat superspace» can be turned, by tensoring it with B_L, into a new, well-behaved sheaf $\mathcal{G}$. Then we introduce the notion of $\mathcal{G}$-supermanifold and prove that the sheaf of derivations on a $\mathcal{G}$-supermanifold is locally free, and that a good graded tangent bundle can be defined. Also the concept of super vector bundle can be naturally introduced; we show that the category of super vector bundles over a $\mathcal{G}$-manifold M is equivalent to the category of locally free graded $\mathcal{G}$-modules. Finally, in Section 5 we demonstrate that $\mathcal{G}$-supermanifolds are a particular case of Rothstein's supermanifolds. In this sense, we prove that any GH^∞ supermanifold can be turned into a Rothstein's supermanifold, even though it is not so by itself.

2. PRELIMINARIES

Let B_L denote the real Grassmann algebra over $\mathbf{R}^L$, $L < \infty$; it has a natural $\mathbf{Z}_2$ gradation $B_L = (B_L)_0 \oplus (B_L)_1$. If $\{e_i : 1 \leq i \leq L\}$ is a basis for $\mathbf{R}^L$, then $e_1, \ldots, e_L$ generate B_L as an algebra, and $\{\beta_\mu \equiv e_{\mu(1)} \wedge \ldots \wedge e_{\mu(r)} : \mu \in \Xi_L\}$ (1) is a real vector space basis for B_L, where $\Xi_L = \cup_{r=1}^{L}\ \mu : \{1, \ldots, r\} \to \{1, \ldots, L\}$ strictly increasing$\}$. Let N_L be the ideal of nilpotents of B_L; then $B_L = \mathbf{R} \oplus N_L$, and the projections $\sigma : B_L \to \mathbf{R}$, $s : B_L \to N_L$ are called *body* and *soul* map respectively.

The cartesian product B_L^{m+n} can be endowed with a structure of graded B_L-module by setting

(1) Henceforth, the wedge product symbol will be omitted.

(2.1) $$B_L^{m+n} = [(B_L)_0^m \times (B_L)_1^n] \oplus [(B_L)_1^m \times (B_L)_0^n] \equiv B_L^{m,n} \oplus B_L^{\bar{m},\bar{n}}.$$

$B_L^{m,n}$ is a $2^{L-1}(m+n)$-dimensional real vector space, and a body map $\sigma^{m,n}$: $: B_L^{m,n} \to \mathbf{R}^m$ is defined by letting $\sigma^{m,n}(x^1 \dots x^m, y^1 \dots y^n) = (\sigma(x^1) \dots \sigma(x^m))$. $B_L^{m,n}$ will be considered as a topological space with its vector space topology.

Any left graded B_L-module Σ can be turned into a right module, and viceversa, by letting

$$xa = (-1)^{|a||x|} ax \qquad \forall \text{ homogeneous } \; x \in \Sigma, \; a \in B_L,$$

where $||$ denotes the grading. Given two graded B_L-modules Σ, Γ, their graded tensor product over B_L can be canonically given a structure of graded B_L-module. We shall always consider $\Sigma \otimes_{B_L} \Gamma$ as endowed with such a structure. A graded B_L-module is said to be *free of rank* (m, n) if it is free of rank $m + n$ over B_L and has based formed by m even and n odd elements.

Our purpose is now to define a sheaf $\mathcal{GH}$ of algebras on $B_L^{m,n}$ so as to introduce GH^∞ supermanifolds as «varieties» modelled on the pair $(B_L^{m,n}, \mathcal{GH})$. Given a smooth manifold X, we denote by $\mathcal{C}_L(W)$ the sections over $W \subset X$ of the sheaf of B_L-valued C^∞ functions on X. Let L and L' be two positive integers, with $L' \leqslant L$, and define for all $U \subset \mathbf{R}^m$ a morphism of graded algebras

$$Z_{L',L} : \mathcal{C}_{L'}(U) \to \mathcal{C}_L((\sigma^{m,0})^{-1}(U))$$

whose explicit expression is (cf. [13])

(2.2) $$Z_{L',L}(f)(x^1 \dots x^m) = \sum_{i_1 \dots i_m = 0}^{L} \frac{1}{i_1! \dots i_m!} (\partial_1^{i_1} \dots \partial_m^{i_m} f)|(\sigma(x^1) \dots \sigma(x^m)) \times \\ \times s(x^1)^{i_1} \dots s(x^m)^{i_m}.$$

It is easily checked that $Z_{L',L}$ is a monomorphism for any U; its image consists of the GH^∞ functions of even variables on $(\sigma^{m,0})^{-1}(U)$.

We define on $(\sigma^{m,n})^{-1}(U)$, where U is open in $\mathbf{R}^m$, the algebra $\mathcal{GH}((\sigma^{m,n})^{-1}(U))$, whose elements have the form:

(2.3) $$F(x^1 \dots x^m, y^1 \dots y^n) = \sum_{\mu \in \Xi_n} F_\mu(x^1 \dots x^m) y^\mu$$

where $y^\mu \equiv y^{\mu(1)} \dots y^{\mu(r)}$ and $F_\mu \in Z_{L',L}(\mathcal{C}_{L'}(U))$. $\mathcal{GH}((\sigma^{m,n})^{-1}(U))$ is naturally equipped with a structure of graded commutative $B_{L'}$-algebra. So we can define a sheaf $\mathcal{GH}$ of graded commutative $B_{L'}$-algebras over $B_L^{m,n}$ by letting, for all open sets $V \subset B_L^{m,n}$,

(2.4) $$\mathcal{GH}(V) = \mathcal{GH}((\sigma^{m,n})^{-1}\sigma^{m,n}(V)).$$

If $L' = L$ we obtain the sheaf of G^∞ functions on $B_L^{m,n}$ [9]. It is known that these functions are badly behaved in many respects [11 - 13], unless $L = 0$ or in the case of only even variables (i.e. $n = 0$). In fact, the odd derivatives are not well defined and, as a consequence, the sheaf of derivations of G^∞ functions is not locally free.

In order to avoid these drawbacks, it is necessary to let $L - L' \geqslant n$. If this condition is verified, which we shall henceforth assume, the sections of the sheaf $\mathcal{GH}$ on $B_L^{m,n}$ are called GH^∞ functions [13]. If F is a GH^∞ function, its derivatives are uniquely determined by the expansion

$$F(z+h) = F(z) + \sum_{A=1}^{m+n} h^A \frac{\partial F}{\partial z^A}(z) + \sum_{A,B=1}^{m+n} h^A h^B g_{AB}(z, h) \tag{2.5}$$

where $z, h \in B_L^{m,n}$. This allows one to prove the sheaf isomorphism

$$\mathcal{GH} \simeq \widehat{\mathcal{GH}} \otimes_{B_{L'}} \Lambda[n] \tag{2.6}$$

where $\widehat{\mathcal{GH}}$ is the subsheaf of $\mathcal{GH}$ whose sections are GH^∞ functions which do not depend on the odd variables, and $\Lambda[n]$ is the exterior algebra over $B_{L'}$ with n generators.

DEFINITION 2.1. A Hausdorff, second countable topological space is an (m, n)-dimensional GH^∞ supermanifold if it admits an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha) \mid \phi_\alpha : U_\alpha \to B_L^{m,n}\}$ such that the transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ are GH^∞ maps. The sheaf of B_L-valued GH^∞ functions on M will be denoted by $\mathcal{GH}^M$, or, when no confusion can arise, simply by $\mathcal{GH}$.

Remarks. (i) The constant GH^∞ functions on a supermanifold are $B_{L'}$-valued. This has the peculiar consequence that the terms in the right hand side of Eq. (2.5), taken one by one, in general are not GH^∞ functions of h.

(ii) A comparison with Rothstein's approach to supermanifolds shows that GH^∞ supermanifolds do not fit into his axiomatics. In this connection see Section 5.

3. ANALYSIS OF THE SHEAF *Der* $\mathcal{GH}$

The purpose of this section is to show that it is not possible to obtain a fully adequate generalization of the category of smooth vector bundles to the context of GH^∞ supermanifolds. In fact, a reasonable definition of «super vector bundle» should yield a category equivalent to the category of locally free graded $\mathcal{GH}$-modules, but this is precluded by the bad behaviour of GH^∞ functions. A particular but important case of this situation is given by the sheaf of graded derivations of the structure sheaf $\mathcal{GH}^M$ of a GH^∞ supermanifold M, which is locally free, and yet does not give rise to a consistent notion of graded tangent space.

Let us introduce the presheaf *Der* $\mathcal{GH}^M$ over M whose sections over an open $U \subset M$ are the morphisms of sheaves of graded $B_{L'}$-algebras $D : \mathcal{GH}^M \mid U \to \to \mathcal{GH}^M \mid U$ which satisfy the graded Leibniz rule, i.e.

$$D(fg) = D(f)g + (-1)^{|f||D|} f D(g) \quad \forall f, g \in \mathcal{GH}^M(V) \text{ for any open } V \subset U.$$

The corresponding sheaf will be again denoted by *Der* $\mathcal{GH}^M$ and its sections will be called *graded derivations* of $\mathcal{GH}^M$.

PROPOSITION 3.1. *The sheaf Der $\mathcal{GH}^M$ is a locally free graded $\mathcal{GH}^M$-module of rank $(m, n) = \dim M$* (2). *In particular, Der $\mathcal{GH}^M(U)$ is the graded $\mathcal{GH}^M(U)$-module generated by*

$$\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\},$$

provided that $(U, (x^1 \ldots x^m, y^1 \ldots y^n))$ is a chart on M.

The proof of the previous proposition is a quite straightforward consequence of the following lemma:

LEMMA 3.1. *Let V be an open set in $B_L^{m,n}$. If $f \in \mathcal{GH}(V)$ is a GH^∞ function of even variables, we have $f = Z_{L',L}(\hat{f})$, with $\hat{f} \in \mathcal{C}_{L'}(\sigma^{m,n}(V))$. Then, for all $D \in \in Der\, \mathcal{GH}(V)$,*

$$Df = Z_{L',L}(\hat{D}\hat{f}),$$

where $\hat{D}$ is the derivation of $\mathcal{C}_{L'}(\sigma^{m,n}(V))$ defined by

$$\hat{D}\hat{g} = [DZ_{L',L}(\hat{g})]_{\sigma^{m,n}(V)} \qquad \forall \hat{g} \in \mathcal{C}_{L'}(\sigma^{m,n}(V)).$$

Proof. For any GH^∞ functions of even variables f_1, f_2 on V, one has $f_1 = f_2$ iff $\hat{f}_1 = \hat{f}_2$, since $Z_{L',L}$ is injective. It is now trivial that $\hat{D}\hat{f} = [Z_{L',L}(\hat{D}\hat{f})]_{\sigma^{m,n}(V)}$, whence the thesis follows. ■

Proof of Proposition 3.1. Since the result to be proven is local, we may assume

(2) We recall that a graded $\mathcal{GH}^M$-module $\mathcal{E}$ is locally free of rank (r, s) if any $x \in M$ has a neighbourhood U such that $\mathcal{E} \mid U$ is isomorphic to $(\mathcal{GH}^M \mid U)^{r+s}$.

$M = B_L^{m,n}$. $Der\ \mathscr{C}_{L'}(\sigma^{m,n}(U))$ is a free $\mathscr{C}_{L'}(\sigma^{m,n}(U))$-module generated by the $\dfrac{\partial}{\partial x^i}$, $i = 1 \dots m$, restricted to $\sigma^{m,n}(U)$, so that, if f is a GH^∞ function of even variables,

$$Df = Z_{L',L}(\hat{D}\hat{f}) = Z_{L',L}\left(\hat{D}(\hat{x}^i)\ \frac{\partial \hat{f}}{\partial \hat{x}^i}\right) = D(x^i)\ \frac{\partial f}{\partial x^i}\ ,$$

and the result is proved in the case of even variables. The thesis now follows from the remark that the coordinate expressions of arbitrary functions in $\mathscr{GH}(U)$ are polynomials in the odd variables. ∎

Despite of its quite good algebraic properties, $Der\ \mathscr{GH}$ has not an intrinsic geometric meaning, for it is not possible to obtain from it, analogously to the smooth case, a graded tangent bundle. More precisely, we might look for a locally trivial GH^∞ fibre bundle $E \xrightarrow{\pi} M$ such that, for all open $U \subset M$, the sections of E over U are the $\mathscr{GH}^M(U)$-module $Der\ \mathscr{GH}^M(U)$; the typical fibre E_x at a point $x \in M$ should be isomorphic to the $B_{L'}$-module $(\mathscr{GH}^M_x/\mathscr{M}_x)^{m+n}$, where $\mathscr{M}_x$ is the maximal ideal of germs of functions vanishing at x (3). But one has $\mathscr{GH}^M_x/\mathscr{M}_x \simeq \mathscr{V}_x$, where

$$\mathscr{V}_x = \{a \in B_L \ \text{ s.t. } \ a = \tilde{f}(x) \ \text{ for some } \ f \in \mathscr{GH}^M_x\}$$

is the graded $B_{L'}$-module of values assumed by all the GH^∞ functions at x (the tilde denotes evaluations of germs). It turns out that $\mathscr{V}_x$ is strictly dependent on the point x, and in general is not free; indeed, in the case $M = B_L^{m,0}$, if x is real, one has $\mathscr{V}_x \equiv B_{L'}$, while, for arbitrary x, $B_{L'} \subset \mathscr{V}_x \subset B_L$. Thus we cannot obtain a graded tangent space at x consistent with the sheaf of derivations, and the possibility of defining E is precluded.

This state of things extends to a more general situation. In fact, the bad behaviour of $\mathscr{V}_x$ is an obstruction to existence of a category of GH^∞ super vector bundles equivalent to the category of locally free $\mathscr{GH}$-modules. This fact can be stressed by trying to construct explicitly the GH^∞ fibre bundle by means of transition functions (see for instance the next section), which cannot be defined since $\mathscr{V}_x$ is not free.

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(3) The isomorphism $E_x \simeq (\mathscr{GH}^M_x/\mathscr{M}_x)^{m+n}$ in the case of smooth or holomorphic vector bundles, which extends a classical result of Serre and Swan [14] valid in the continuous case, is easily proved by taking into account the explicit relationship between a locally free sheaf and its associated vector bundle.

4. $\mathscr{G}$-SUPERMANIFOLDS AND $\mathscr{G}$-SUPER VECTOR BUNDLES

In this section we shall show that a satisfactory theory of supermanifolds, where a well-behaved tangent bundle can be introduced, is achieved by considering a new sheaf $\mathscr{G}$ as the relevant structure sheaf. In this way one also gets a good theory of super vector bundles; indeed, we shall show that the category of super vector bundles is equivalent to the category of locally free graded $\mathscr{G}$-modules.

Let $\mathscr{G}\mathscr{H}$ be the sheaf of GH^∞ functions over $B_L^{m,n}$, and define

$$\mathscr{G} = \mathscr{G}\mathscr{H} \otimes_{B_{L'}} B_L. \tag{4.1}$$

By introducing the multiplication $(f \otimes a)(g \otimes b) = (-1)^{|a||g|} fg \otimes ab$, $\mathscr{G}$ becomes a sheaf of graded commutative B_L-algebras. Let $\delta : \mathscr{G} \to \mathscr{C}_L$, where $\mathscr{C}_L$ is the sheaf of smooth B_L-valued functions on $B_L^{m,n}$, be the «evaluation» morphism given by

$$\delta(f \otimes a) = fa. \tag{4.2}$$

The image of δ is the sheaf $\mathscr{G}^\infty$ of G^∞ functions on $B_L^{m,n}$ (see Section 2), while, on the other hand, δ is injective only if restricted to the subsheaf

$$\hat{\mathscr{G}} = \widehat{\mathscr{G}\mathscr{H}} \otimes_{B_{L'}} B_L.$$

The isomorphism $\delta : \hat{\mathscr{G}} \to \hat{\mathscr{G}}^\infty$ is proved by exhibiting a map $\eta : \hat{\mathscr{G}}^\infty \to \hat{\mathscr{G}}$ such that $\eta \circ \delta = \delta \circ \eta = id$. Given an open set $U \subset B_L^{m,n}$, any $f \in \hat{\mathscr{G}}^\infty(U)$ can be written as

$$f = Z_{0,L}(\hat{f}^\mu)\beta_\mu \tag{4.3}$$

where the $\hat{f}^\mu$ are smooth real-valued functions on $\sigma^{m,n}(U)$. Then we set $\eta(f) = Z_{0,L}(\hat{f}^\mu) \otimes \beta_\mu$.

Now, if U is a connected subset of $B_L^{m,n}$, the set of sections $f \in \mathscr{G}(U)$ which are constant, in the sense that $\delta(f)$ is constant, is isomorphic with B_L. The germs in $\mathscr{G}_x$ can be evaluated by composing δ with the evaluation of a G^∞ function at x, thus obtaining a surjective map $\sim : \mathscr{G}_x \to B_L$; then we have an exact sequence of graded B_L-modules

$$0 \to \mathscr{M}_x \to \mathscr{G}_x \to B_L \to 0. \tag{4.4}$$

The sheaf $Der\,\mathscr{G}$ of graded derivations of $\mathscr{G}$ is defined in analogy with $Der\,\mathscr{G}\mathscr{H}$.

PROPOSITION 4.1. *The sheaf $Der\,\mathscr{G}$ is isomorphic to the sheaf $Der\,\mathscr{G}\mathscr{H} \otimes_{B_{L'}} B_L$.*

Proof. We first prove the isomorphism $Der\,\hat{\mathscr{G}} \simeq Der\,\widehat{\mathscr{G}\mathscr{H}} \otimes_{B_{L'}} B_L$. Identifying $\hat{\mathscr{G}}$ with $\hat{\mathscr{G}}^\infty$, we define a map $\nu : Der\,\hat{\mathscr{G}} \to Der\,\widehat{\mathscr{G}\mathscr{H}} \otimes_{B_{L'}} B_L$ by setting (with reference to Eq. (4.3))

$$\nu(D)(f) = D(Z_{0,L}(\hat{f}^\mu)) \otimes \beta_\mu .$$

It is easily shown that ν is an isomorphism; then the thesis is a consequence of the isomorphism $\mathcal{G} \simeq \hat{\mathcal{G}} \otimes_{B_{L'}} \Lambda[n]$, which follows from Eq. (2.6). ■

Propositions 3.1 and 4.1 imply that *Der* $\mathcal{G}$ it is a locally free graded $\mathcal{G}$-module of rank (m, n). In particular, if U is an open set in $B_L^{m,n}$, *Der* $\mathcal{G}(U)$ is the graded $\mathcal{G}(U)$-module generated by the derivations

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right\},$$

defined as

$$\frac{\partial}{\partial x^i}(f \otimes a) = \frac{\partial f}{\partial x^i} \otimes a, \quad i = 1 \ldots m;$$

$$\frac{\partial}{\partial y^\alpha}(f \otimes a) = \frac{\partial f}{\partial y^\alpha} \otimes a, \quad \alpha = 1 \ldots n. \tag{4.5}$$

We wish now to introduce $\mathcal{G}$-supermanifolds as manifolds modelled on the pair $(B_L^{m,n}, \mathcal{G})$. If U and V are open sets in $B_L^{m,n}$, a smooth map $\varphi : U \to V$ is said to be a $\mathcal{G}$*-map* if $\varphi^*(\mathcal{G} \mid V)$ is a subsheaf of $\mathcal{G} \mid U$.

DEFINITION 4.1. A Hausdorff, second countable topological space M is an (m, n)-dimensional $\mathcal{G}$-supermanifold if it admits an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha) \mid \phi_\alpha : U_\alpha \to \to B_L^{m,n}$ such that the transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ are $\mathcal{G}$-maps. The structure sheaf $\mathcal{G}^M$ is by definition the sheaf on M such that $\phi_\alpha^* : \mathcal{G} \mid \phi_\alpha(U_\alpha) \to \mathcal{G}^M \mid U_\alpha$ is a sheaf isomorphism for any α. When no confusion can arise, $\mathcal{G}^M$ will be denoted simply by $\mathcal{G}$.

A GH^∞ supermanifold M can be turned into a $\mathcal{G}$-supermanifold, having the same transition functions, whose structure sheaf satisfies the condition

$$\mathcal{G}^M \simeq \mathcal{GH}^M \otimes_{B_{L'}} B_L . \tag{4.6}$$

On the other hand, it should be noticed that Eq. (4.6) is not always true, since $\mathcal{G}$-maps are not GH^∞ maps. Thus, the category of GH^∞ supermanifolds is strictly included into the category of $\mathcal{G}$-supermanifolds.

However, it is a trivial consequence of Proposition 4.1 that *Der* $\mathcal{G}^M$ is a locally free graded $\mathcal{G}$-module of rank (m, n), where $(m, n) = \dim M$. The graded $\mathcal{G}^M(U)$-module basis of *Der* $\mathcal{G}^M$ is given by the derivations $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right)$ defined in Eq. (4.5), provided that $(U, (x^1 \ldots x^m, y^1 \ldots y^n))$ is a chart on M.

DEFINITION 4.2. A $\mathcal{G}$-superbundle is a triple (E, M, π), where E and M are $\mathcal{G}$-supermanifolds and π is a surjective $\mathcal{G}$-map. (E, M, π) is said to be locally trivial with standard fibre F, where F is a $\mathcal{G}$-supermanifold, if M admits a cover $\{U_\alpha\}$ with $\mathcal{G}$-diffeomorphisms

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F \qquad \text{such that } \; pr_2 \circ \psi_\alpha = \pi.$$

Finally, if F is a free graded B_L-module of rank (r, s), and the maps ψ_α restricted to the fibres $\pi^{-1}(x)$ are isomorphisms of graded B_L-modules, (E, M, π) is said to be a $\mathcal{G}$-super vector bundle of rank (r, s).

In the following, $\mathcal{G}$-super vector bundles will be referred to simply as «SVB's». Given two SVB's E, E' over M, a morphism $\varphi : E \to E'$ is a $\mathcal{G}$-map which, restricted to the fibres of E, yields morphisms of graded B_L-modules into the fibres of E'. The collection of isomorphism classes of all SVB's of rank (r, s) over M, together with the morphisms of SVB's over M, constitutes a category, that we denote by $\mathbf{SVB}_{(r,s)}(M)$.

We wish now to show the equivalence between $\mathbf{SVB}_{(r,s)}(M)$ and the category of isomorphism classes of locally free $\mathcal{G}^M$-modules over M. This is most easily shown by using transition functions to specify the bundle. Given an SVB (E, M, π) of rank (r, s) with standard fibre F, we identify F with B_L^{r+s}. After fixing trivializing isomorphisms $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times B_L^{r+s}$, we can construct transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, s)$, where $GL(r, s)$ is the super Lie group of even automorphisms of B_L^{r+s} as a graded B_L-module [15], by setting, for $x \in U_\alpha \cap U_\beta$,

$$g_{\alpha\beta}(x)(\,\cdot\,) = pr_2 \circ \psi_\alpha \circ \psi_\beta^{-1}(x, \cdot).$$

These transition functions fulfil the usual cocycle condition

$$g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = 1 \qquad \forall\, x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{4.7}$$

A standard argument [16] shows that a set of $\mathcal{G}$-maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, s)$ satisfying the cocycle condition (4.7) determines an equivalence class of SVB's on M whose representatives have the given $g_{\alpha\beta}$'s as transition functions.

Let $\mathcal{E}$ be a locally free $\mathcal{G}$-module of rank (r, s); then on M there is a cover $\{U_\alpha\}$ together with a collection of isomorphisms

$$\varphi_\alpha : \mathcal{E}| \, U_\alpha \to (\mathcal{G}^M \mid U_\alpha)^{r+s}.$$

Now we define sheaf morphisms

$$h_{\alpha\beta} : (\mathcal{G}^M \mid U_\alpha \cap U_\beta)^{r+s} \to (\mathcal{G}^M \mid U_\alpha \cap U_\beta)^{r+s}$$

by setting $h_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$. Thus we obtain $\mathcal{G}$-maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, s)$ whose value at $x \in U_\alpha \cap U_\beta$ is fixed by the requirement

$$\widetilde{h_{\alpha\beta}(f)} = g_{\alpha\beta}(x) \cdot \tilde{f}$$

for all $f \in \mathcal{G}_x^{r+s}$. These maps satisfy the cocycle condition (4.7), and therefore give rise to an isomorphism class of SVB's over M. If E is any SVB in this isomorphism class, we have a canonical isomorphism $E_x \cong \mathcal{E}_x/\mathcal{M}_x\mathcal{E}_x$ ($\mathcal{M}_x$ was defined in Eq. (4.4)).

Conversely, the sheaf of sections of an SVB E over M is easily shown to be a locally free graded $\mathcal{G}^M$-module. Since the two processes are one the inverse of the other, and are well-behaved with respect to the morphisms, we end up with the following result:

PROPOSITION 4.2. $\mathbf{SVB}_{(r,s)}(M)$ *and the category of isomorphism classes of locally free graded* $\mathcal{G}^M$*-modules of rank* (r, s) *over* M *are equivalent.* ■

In particular, the sheaf $Der\,\mathcal{G}^M$ corresponds to a rank (m, n) SVB (where $(m, n) = \dim M$) that we call the *graded tangent bundle* to M and denote by TM; its fibre T_xM is called the *graded tangent space* to M at x. The next two results will show that TM is a genuine generalization of the ordinary tangent bundle to a smooth manifold.

PROPOSITION 4.3. T_xM *is canonically isomorphic to the graded* B_L*-module* $\mathcal{D}_x$ *of morphisms of graded* B_L*-modules* $X: \mathcal{G}_x^M \to B_L$ *satisfying the graded Leibniz rule*

$$X(fg) = X(f)\tilde{g} + (-1)^{|f||X|}\tilde{f}X(g) \qquad \forall f, g \in \mathcal{G}_x^M. \tag{4.8}$$

Proof. Regarding T_xM as the quotient $(Der\,\mathcal{G}^M)_x/\mathcal{M}_x(Der\,\mathcal{G}^M)_x$, we establish a map $T_xM \to \mathcal{D}_x$ by letting

$$\bar{D} \mapsto X, \qquad X(f) = \widetilde{Df},$$

where $D \in (Der\,\mathcal{G}^M)_x$ is any representative of $\bar{D} \in T_xM$. By explicit computation, one can verify that X fulfils the graded Leibniz rule (4.8), and that this map is an isomorphism. ■

COROLLARY 4.1. *The even part of* T_xM *is canonically isomorphic, as a real vector space, to the ordinary tangent space at* x *to the smooth manifold underlying* M.

Proof. This can be proved as in Proposition 1.8 of Ref. [12]. ■

5. COMPARISON WITH ROTHSTEIN'S SUPERMANIFOLDS

Now we compare our approach to supermanifolds with the work of Rothstein [12], who has formulated a set of axioms to characterize supermanifolds; one of these is that the sheaf of graded derivations is locally free. Rothstein's axiomatics entails the existence of a good graded tangent space, which – in the case that the graded commutative algebra underlying the theory is B_L – is a free graded B_L-module.

We start by reviewing Rothstein's axiomatics. Let B be a graded commutative Banach $\mathbf{R}$-algebra, M a Hausdorff topological space, $\mathscr{A}$ a sheaf of graded commutative B-algebras with base M, and finally let δ be a morphism of sheaves of graded commutative algebras from $\mathscr{A}$ into the sheaf $\mathscr{C}^0_B$ of B-valued continuous functions on M. We say that the triple $(M, \mathscr{A}, \delta)$ is a *Rothstein's supermanifold* of dimension (m, n) if and only if:

A1. There exist coordinate charts $(U, (x^1 \ldots x^m, y^1 \ldots y^n))$ on M, i.e. the U's are open sets which cover M, and (x^i, y^α) are sections of $\mathscr{A}(U)$ such that $(\mathrm{d}x^1 \ldots \mathrm{d}x^m, \mathrm{d}y^1 \ldots \mathrm{d}y^n)$ is a graded $\mathscr{A}(U)$-basis for $Der^*\mathscr{A}(U)$.

A2. The functions $(\delta(x^i), \delta(y^\alpha))$ give a homeomorphism of U into $B^{m,n}$.

A3. For any $x \in U$, and $f \in \mathscr{A}_x$, there exist $g_1 \ldots g_{m+n} \in \mathscr{A}_x$ such that

$$f = \delta(f)(x) + \sum_{i=1}^{m} g_i(x^i - \delta(x^i)(x)) + \sum_{\alpha=1}^{n} g_{m+\alpha}(y^\alpha - \delta(y^\alpha)(x)). \tag{5.1}$$

A4. If for all differential operators K on $\mathscr{A}$ an $f \in \mathscr{A}_x$ satisfies $\delta(Kf) = 0$, then $f = 0$.

Remark. Axiom A3 means that the evaluations of sections of $\mathscr{A}$ representing the l.h.s. and r.h.s. of (5.1) in any point $y \in U$ sufficiently close to x give the same result, where the «evaluation of h at y» is $\delta(h)(y)$.

Since GH^∞ supermanifolds do not have a good tangent bundle, they cannot verify these axioms. In fact, if we set $B = B_{L'}$, axiom A1 holds (this is our Proposition 3.1), but there is no way to define an evaluation map δ such as to satisfy axiom A2. Indeed, the most reasonable choice would be $\delta(f)(x) = p_{LL'}(f(x))$ $\forall f \in \mathscr{G}\mathscr{H}^M(U)$, $\forall x \in U \subset M$, where $p_{LL'} : B_L \to B_{L'}$ is the natural projection; this map δ obviously is neither suitable to satisfy axiom A2, nor axiom A3, even though axiom A4 is verified.

On the contrary, $\mathscr{G}$-supermanifolds fit into this axiomatics. Indeed, let $B = B_L$, $\mathscr{A} = \mathscr{G}$, and take δ as defined in Eq. (4.2). Axiom A1 is a direct consequence of Proposition 4.1, while axiom A2 follows from Definition 4.1. We check axioms

A3 and A4 taking $M = B_L^{m,n}$, which is allowed since the statements are of a local kind. By virtue of the Taylor expansion (2.5) for GH^∞ functions, we have that, if U is an open set in $B_L^{m,n}$, and $z_1, z_2 \in U$, the equation

$$\begin{aligned}\delta(f\otimes a)(z_2) &= \delta(f\otimes a)(z_1) + (\delta(x^i)(z_2) - \delta(x^i)(z_1))\cdot\delta\left(\frac{\partial f}{\partial x^i}\right)\otimes a\ (z_1)\\ &+ (\delta(y^\alpha)(z_2) - \delta(y^\alpha)(z_1))\cdot\delta\left(\frac{\partial f}{\partial y^\alpha}\otimes a\right)(z_1) + O(\|z_2 - z_1\|^2)\end{aligned} \tag{5.2}$$

hods for any $f\otimes a \in \mathcal{G}(U)$ provided that z_2 and z_1 are close enough. Eq. (5.2) is equivalent to Eq. (5.1), so that axiom A3 is valid for $\mathcal{G}$-functions. In order to prove axiom A4, we recall that on the even part of $\mathcal{G}(U)$ the evaluation map δ is injective, so that, if $\delta(Kf) = 0$ for all differential operators K, in particular $\delta(f) = 0$, and then $f = 0$ if $f \in \hat{\mathcal{G}}(U)$. Since a generic $\mathcal{G}$-function is a polynomial in the odd variables, axiom A4 follows.

The previous discussion can be summarized in the following result.

PROPOSITION 5.1. *A $\mathcal{G}$-supermanifold is a Rothstein's supermanifold.* ■

CONCLUDING REMARK

In two previous papers [17 - 18] we have studied the cohomology of GH^∞ supermanifolds; in particular, we have analyzed the Cech cohomology of the structure sheaf $\mathcal{GH}^M$ of a GH^∞ supermanifold M, and a generalization of the de Rham cohomology, which consists in the cohomology of the complex of global GH^∞ differential forms on M. The results there established can be readily transferred to the context of $\mathcal{G}$-supermanifolds.

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